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THERMAL STRESSES IN A TRANSVERSELY ISOTROPIC ELASTIC SOLID WEAKENED BY AN EXTERNAL CIRCULAR CRACK

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Abstract—This paper examines the problem of the steady state thermoelastic behaviour of an external circular crack located in a transversely isotropic elastic medium. The surfaces of the crack are subjected to a symmetric temperature distribution. The mathematical analysis of the problem is approached via a Hankel transform development of the governing equations. Numerical results presented in the paper illustrate the manner in which the thermoelastic stress intensity factors are governed by the elastic and thermal properties of the transversely isotropic elastic solid.

1. INTRODUCTION

The problem of determining the thermal stresses and stress intensity factors in elastic media containing cracks or planes of discontinuities has attracted considerable attention. The results of such investigations have important engineering application in the study of fracture mechanics of structural components subjected to adverse environmental effects. The earlier studies of thermoelastic behaviour of cracks are due to Olesiak and Sneddon[1]. They examined the steady state thermoelastic behaviour of a penny-shaped crack, the surfaces of which are subjected to a prescribed heat flux. A Hankel transform technique is used to formulate the mixed boundary value problem as a set of dual integral equations. The case in which the surface of the penny-shaped crack is subjected to antisymmetric heat fluxes was examined by Florence and Goodier[2]. Kassir and Sih[3] and Kassir[4] considered the thermoelastic external crack problem for an isotropic elastic material calculated the stress intensity factor and determined the extent of the plastic zone around the leading edge of the crack. The Dugdale type plastic behaviour approximately models the ductile behaviour of the material. More recently, Tsai[5,6] examined the thermoelastic problem of a penny-shaped crack in a transversely isotropic medium by employing Hankel transform techniques and double integration techniques.

In this paper we examine the thermoelastic problem for an external circular crack which is located in a transversely isotropic elastic solid. The principal axis of transverse isotropy (for both plastic and thermal responses) lies normal to the plane of the external crack. The temperature on the faces of the crack are assumed to be nonuniform. Again, the mathematical analysis of the problem is approached by adopting a Hankel transform development of the problem. Numerical results are developed to illustrate the manner in which the stress intensity factor for the crack is influenced by the elastic and thermal properties of the transversely isotropic material.

2. AXISYMMETRIC EQUATIONS OF THERMOELASTICITY

Consider an infinite transversely isotropic elastic solid containing an external circular crack of unit radius in a plane of the solid. In cylindrical polar coordinates (r, θ, z), let the faces of the crack be assumed as $r > 1$, $z = 0^{\pm}$ with the origin of the coordinate system at

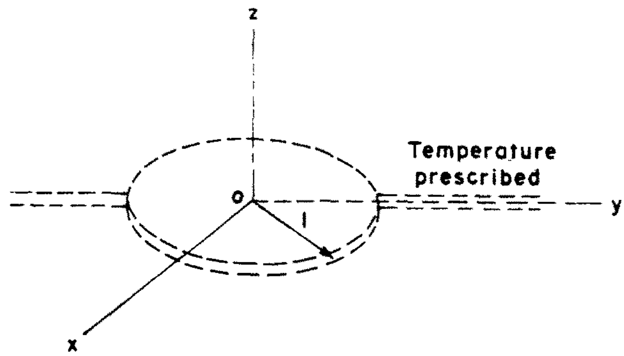

Fig. 1. Geometry of external circular crack.

the centre of the crack. It is assumed that the crack opens symmetrically by the application of a temperature distribution. Since the thermal and mechanical conditions on the crack exhibit a state of symmetry about the plane $z = 0$, we may restrict the discussion to the study of a single halfspace region $0 < z < \infty$, where the bounding smooth plane surface $z = 0$ is subjected to appropriate mixed boundary conditions. The displacement vector will have the components $(u, 0, w)$ and the non-vanishing components of the stress tensor will be σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz}. The geometry of the crack problem is shown in Fig. 1.

Under steady state conditions, the temperature field $T = T(r, z)$ at any point satisfies the steady state heat conduction equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \beta^2 \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

where $\beta^2 = k_z/k_r$ is the ratio of the coefficients of thermal conductivity along the z-axis and in the z-plane. In terms of displacements, the equations of equilibrium for a transversely isotropic material are given by

$$c_{11}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + c_{44}\frac{\partial^2 u}{\partial z^2} + (c_{13} + c_{44})\frac{\partial^2 w}{\partial r \partial z} = b_1 \frac{\partial T}{\partial r} \tag{2}$$

$$c_{44}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + c_{33}\frac{\partial^2 w}{\partial z^2} + (c_{13} + c_{44})\left(\frac{\partial w}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) = b_2 \frac{\partial T}{\partial z} \tag{3}$$

where

$$b_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_2, \qquad b_2 = 2\alpha_1 c_{13} + c_{33}\alpha_2 \tag{4}$$

and α_1 and α_2 are the coefficients of linear thermal expansion along and perpendicular to the z-axis. Equations (1)–(3) are to be solved, subject to the following boundary conditions:

$$\begin{aligned} &\frac{\partial T}{\partial z} = 0, \qquad 0 \leq r < 1, \qquad z = 0, \\ &T = T(r), \qquad r > 1, \qquad z = 0. \end{aligned} \tag{5}$$

and

$$\begin{aligned} w(r,0) &= 0, && 0 \leqslant r < 1, \\ \sigma_z(r,0) &= -\sigma(r), && r > 1, \\ \sigma_{rz}(r,0) &= 0, && 0 \leqslant r < \infty. \end{aligned} \tag{6}$$

Also, the displacements and stresses derived from the analysis should reduce to zero as r, $z \to \infty$. Also the functions $T(r)$ and $\sigma(r)$ appearing in boundary conditions (5) and (6), respectively, must be bounded at infinity.

3. THE STEADY STATE TEMPERATURE DISTRIBUTION

The Hankel transform solution of eqn (1) takes the form

$$T(r,z) = \int_0^\infty A(\xi)\,\mathrm{e}^{-\xi z/\beta} J_0(\xi r)\,\mathrm{d}\xi \tag{7}$$

where $J_0(\xi r)$ is the Bessel function of the first kind of order zero and $A(\xi)$ is an unknown function of ξ which should be determined by satisfying boundary conditions (5). Equation (7), with the aid of boundary conditions (5), gives a set of dual integral equations

$$\int_0^\infty \xi A(\xi) J_0(\xi r)\,\mathrm{d}\xi = 0, \qquad 0 \leqslant r < 1 \tag{8}$$

$$\int_0^\infty A(\xi) J_0(\xi r)\,\mathrm{d}\xi = T_0 g(r), \qquad 1 < r. \tag{9}$$

The solution of the dual integral equations, eqns (8) and (9), is given by Lowengrub and Sneddon[7]. Assume that $A(\xi)$ admits a representation of the form

$$A(\xi) = -\int_1^\infty \sin(\xi t)\,\phi'(t)\,\mathrm{d}t \tag{10}$$

where

$$\phi(t) = \frac{2}{\pi} T_0 \int_t^\infty \frac{r g(r)\,\mathrm{d}r}{(r^2 - t^2)^{1/2}} \tag{11}$$

$$\phi'(t) = \frac{\mathrm{d}\phi}{\mathrm{d}t}. \tag{12}$$

The expression for the temperature function can be written as

$$T(r,z) = -\int_1^\infty \phi'(t)\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-\xi z/\beta} J_0(\xi r) \sin(\xi t)\,\mathrm{d}\xi. \tag{13}$$

From eqn (13) we find that

$$T(r,0) = -\int_{\max(1,r)}^{\infty} \frac{\phi'(t)\,\mathrm{d}t}{(t^2 - r^2)^{1/2}}. \tag{14}$$

Making use of the analysis given in the appendix we can obtain the displacements and stresses at any point in the following forms:

$$w(r,z) = \int_0^\infty \left[\frac{\lambda_1}{m_1}\xi B(\xi)\,\mathrm{e}^{-\xi z/m_1} + \frac{\lambda_2}{m_2}\xi C(\xi)\,\mathrm{e}^{-\xi z/m_2} + \frac{\mu_2}{\beta}\xi^{-1}A(\xi)\,\mathrm{e}^{-\xi z/\beta}\right] J_0(\xi r)\,\mathrm{d}\xi$$

$$\sigma_{zz}(r,z) = \int_0^\infty \left[\left(\frac{\lambda_1}{m_1^2}c_{33} - c_{13}\right)\xi^2 B(\xi)\,\mathrm{e}^{-\xi z/m_1} + \left(\frac{\lambda_2}{m_2^2}c_{33} - c_{13}\right)\xi^2 C(\xi)\,\mathrm{e}^{-\xi z/m_2}\right.$$
$$\left. + \left(\frac{\mu_2}{\beta^2}c_{33} - \mu_1 c_{13} - b_2\right)A(\xi)\,\mathrm{e}^{-\xi z/\beta}\right] J_0(\xi r)\,\mathrm{d}\xi$$

$$\sigma_{rz}(r,z) = c_{44}\int_0^\infty \left[\frac{1+\lambda_1}{m_1}\xi^2 B(\xi)\,\mathrm{e}^{-\xi z/m_1} + \frac{1+\lambda_2}{m_2}\xi^2 C(\xi)\,\mathrm{e}^{-\xi z/m_2}\right.$$
$$\left. + \frac{\mu_1+\mu_2}{\beta}A(\xi)\,\mathrm{e}^{-\xi z/\beta}\right] J_1(\xi r)\,\mathrm{d}\xi. \tag{15}$$

With the help of boundary condition $(6)_3$ and eqn $(15)_3$ we get

$$B(\xi) = -\frac{\left(\dfrac{1+\lambda_2}{m_2}\right)C(\xi) + \left(\dfrac{\mu_1+\mu_2}{\beta}\right)A(\xi)\xi^{-2}}{\left(\dfrac{1+\lambda_1}{m_1}\right)}. \tag{16}$$

Using eqns $(15)_1$ and $(15)_2$ we can write boundary conditions $(6)_1$ and $(6)_2$ as

$$\int_0^\infty \left[\frac{N_1 A(\xi)}{\xi} + \frac{N_2 D(\xi)}{\xi}\right] J_0(\xi r)\,\mathrm{d}\xi = 0, \qquad 0 < r < 1 \tag{17}$$

$$\int_0^\infty [N_3 A(\xi) + N_4 D(\xi)] J_0(\xi r)\,\mathrm{d}\xi = -\sigma(r), \qquad 1 < r \tag{18}$$

where

$$\xi^2 C(\xi) = D(\xi),$$

$$N_1 = \frac{\lambda_2}{\beta} - \frac{\lambda_1(\mu_1+\mu_2)}{\beta(1+\lambda_1)},$$

$$N_2 = \frac{\lambda_2}{m_2} - \frac{\lambda_1(1+\lambda_2)}{m_2(1+\lambda_1)},$$

$$N_3 = \frac{\mu_2}{\beta^2}c_{33} - b_2 - \mu_1 c_{13} - \frac{m_1}{(1+\lambda_1)}\left(\frac{\mu_1+\mu_2}{\beta}\right)\left(\frac{\lambda_1}{m_1^2}c_{33} - c_{13}\right),$$

$$N_4 = \frac{\lambda_2}{m_2^2}c_{33} - c_{13} - \frac{m_1(1+\lambda_2)}{m_2(1+\lambda_1)}\left(\frac{\lambda_1}{m_1^2}c_{33} - c_{13}\right) \tag{19}$$

where $A(\xi)$ is known from eqns (10) and (11). $D(\xi)$ is an unknown function to be determined from eqns (17) and (18). Equation (10) can be rewritten as

$$A(\xi) = \phi(1)\sin(\xi) + \xi\int_1^\infty \phi(t)\cos(\xi t)\,\mathrm{d}t. \tag{20}$$

Making use of eqns (10) and (20), we can write dual integral equations, eqns (17) and (18), as

$$\int_0^\infty \frac{D(\xi)}{\xi} J_0(\xi r)\,\mathrm{d}\xi = f(r), \qquad 0 < r < 1 \tag{21}$$

$$\int_0^\infty D(\xi) J_0(\xi r)\,\mathrm{d}\xi = g_2(r), \qquad r > 1 \tag{22}$$

where

$$f(r) = \frac{-\pi N_1}{2N_2}\phi(1) \tag{23}$$

$$\begin{aligned} g_2(r) &= \frac{-\sigma(r)}{N_4} + \left(\frac{N_3}{N_4}\right)\int_r^\infty \frac{\phi'(t)\,\mathrm{d}t}{(t^2 - r^2)^{1/2}} \\ &= \frac{-\sigma(r)}{N_4} - \frac{N_3}{N_4} T(r, 0), \qquad r > 1 \end{aligned} \tag{24}$$

where the prime denotes the derivative with respect to t.

From Noble[8], the solution of the dual integral equations, eqns (21) and (22), can be written as

$$D(\xi) = \frac{2}{\pi}\xi\left[\int_0^1 F(x)\cos(x\xi)\,\mathrm{d}x + \int_1^\infty G(x)\cos(x\xi)\,\mathrm{d}x\right] \tag{25}$$

where

$$F(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{rf(r)\,\mathrm{d}r}{(x^2 - r^2)^{1/2}} \tag{26}$$

and

$$G(x) = \int_x^\infty \frac{rg_2(r)\,\mathrm{d}r}{(r^2 - x^2)^{1/2}}. \tag{27}$$

Using integration by parts, expression (25) can be rewritten as

$$D(\xi) = \frac{2}{\pi}\left[F(1)\sin(\xi) - G(1)\sin\xi - \int_0^1 F'(x)\sin(x\xi)\,\mathrm{d}x - \int_1^\infty G'(x)\sin(x\xi)\,\mathrm{d}x\right] \tag{28}$$

we find that

$$\sigma_z(r,0) = \left[N_3\int_0^\infty A(\xi)J_0(\xi r)\,d\xi + N_4\int_4^\infty D(\xi)J_0(\xi r)\,d\xi\right]. \tag{29}$$

Making use of eqns (10), (28) and (29) we have

$$\sigma_z(r,0) = \frac{2}{\pi}N_4\left[\frac{F(1)-G(1)}{\sqrt{(1-r^2)}} - \int_r^1 \frac{F'(x)\,dx}{\sqrt{(x^2-r^2)}} - \int_1^\infty \frac{G'(x)\,dx}{\sqrt{(x^2-r^2)}}\right]$$
$$- N_3\int_1^\infty \frac{\phi'(t)\,dt}{\sqrt{(t^2-r^2)}}, \qquad 0<r<1. \tag{30}$$

The expression for the stress intensity factor can be written as

$$k_1 = \lim_{r\to 1^+}[2(1-r)]^{1/2}\sigma_z(r,0) \tag{31}$$

and with the help of eqns (30) and (31) we get

$$k_1 = \frac{2}{\pi}N_4[F(1)-G(1)]. \tag{32}$$

It follows that

$$w(r,0) = \left[N_1\phi(1)\operatorname{cosec}^{-1}(r) + N_1\int_1^r \frac{\phi(t)\,dt}{(r^2-t^2)^{1/2}}\right.$$
$$\left. + \frac{2}{\pi}N_2\int_0^1 \frac{F(t)\,dt}{(r^2-t^2)^{1/2}} + \frac{2}{\pi}N_2\int_1^r \frac{G(t)\,dt}{(r^2-t^2)^{1/2}}\right], \qquad 1<r. \tag{33}$$

4. CRACK SUBJECTED TO NON-UNIFORM TEMPERATURES

If the temperature variation on the crack faces is such that

$$T(r,0) = T_0 g(r), \qquad g(r) = r^{-n}, \qquad n>1, \qquad r>1 \tag{34}$$

then eqn (11) yields

$$\phi(t) = \frac{T_0}{\sqrt{\pi}}\frac{\Gamma(n/2-1/2)t^{1-n}}{\Gamma(n/2)}, \qquad n>1 \tag{35}$$

where $\Gamma(n)$ is the Gamma function. Putting $\phi(t)$ into eqn (14) and carrying out the integration, $T(r,0)$ is obtained in the form

$$T(r,0) = \frac{T_0(n-1)\Gamma(n/2-1/2)}{2\sqrt{\pi}\Gamma(n/2)}r^{-n}B_{r^2}(n/2,1/2), \qquad 0<r<1, \qquad n>1 \tag{36}$$

where $B_x(m,n)$ is the incomplete Beta function defined by

$$B_x(m,n) = \int_0^x y^{m-1}(1-y)^{n-1}\,dy, \qquad \operatorname{Re}[m]>0, \qquad \operatorname{Re}[n]>0. \tag{37}$$

We find that

$$F(x) = -\frac{\pi}{2}\frac{N_1}{N_2}\phi(1). \tag{38}$$

If $\sigma(r) = 0$, then making use of eqns (24), (27) and (35) we get

$$2N_4 \overline{x^{n-}} \quad \overline{\Gamma(n/2)}$$

Making use of eqns (35), (38) and (39), we find from eqn (30) that

$$\sigma_z(r,0) = \frac{T_0 N_4 \Gamma((n-1)/2)}{\Gamma(n/2)\sqrt{(\pi(1-r^2))}}\left[\frac{N_3}{N_4} - \frac{N_1}{N_2}\right]. \tag{40}$$

Using eqns (32), (35), (38) and (39), the stress intensity factor can be written as

$$k_1 = \frac{T_0 N_4 \Gamma((n-1)/2)}{\sqrt{\pi}\Gamma(n/2)}\left[\frac{N_3}{N_4} - \frac{N_1}{N_2}\right]. \tag{41}$$

The expression for the displacement component can be written as

$$w(r,0) = \frac{T_0}{2\sqrt{\pi}}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)} N_2\left[\frac{N_1}{N_2} - \frac{N_3}{N_4}\right] r^{1-n} \times B_{1-r^{-2}}\left[\frac{1}{2}, 1 - \left(\frac{n}{2}\right)\right],$$

$$1 < r. \tag{42}$$

The values of the elastic constants $(c_{11}, c_{12}, c_{13}, c_{33}, c_{44})$, the coefficient of linear expansion (α_1, α_2) and the ratio β^2 of the coefficients of cadmium material are given as follows:

$c_{11} = 11 \times 10^{10}$, $c_{12} = 4.04 \times 10^{10}$, $c_{13} = 3.83 \times 10^{10}$,
$c_{33} = 4.69 \times 10^{10}$, $c_{44} = 1.56 \times 10^{10}$, $\alpha_1 = 54 \times 10^{-5}$,
$\alpha_2 = 20.2 \times 10^{-5}$, $\beta^2 = 1$.

The elastic constants for magnesium are given as follows:

$c_{11} = 5.97 \times 10^{10}$, $c_{12} = 2.62 \times 10^{10}$, $c_{13} = 2.17 \times 10^{10}$,
$c_{44} = 1.64 \times 10^{10}$, $\alpha_1 = 27.7 \times 10^{-5}$, $\alpha_2 = 20.2 \times 10^{-5}$,
$\beta^2 = 1$.

The elastic constants c_{ij} are in units of $\mathrm{N\,m^{-2}}$ and α_i is in $(^\circ\mathrm{C})^{-1}$. The variation of the stress intensity factor with temperature distribution defined by eqn (34) is shown in Fig. 2. As temperature decreases, the stress intensity factor decreases for both cadmium and magnesium materials.

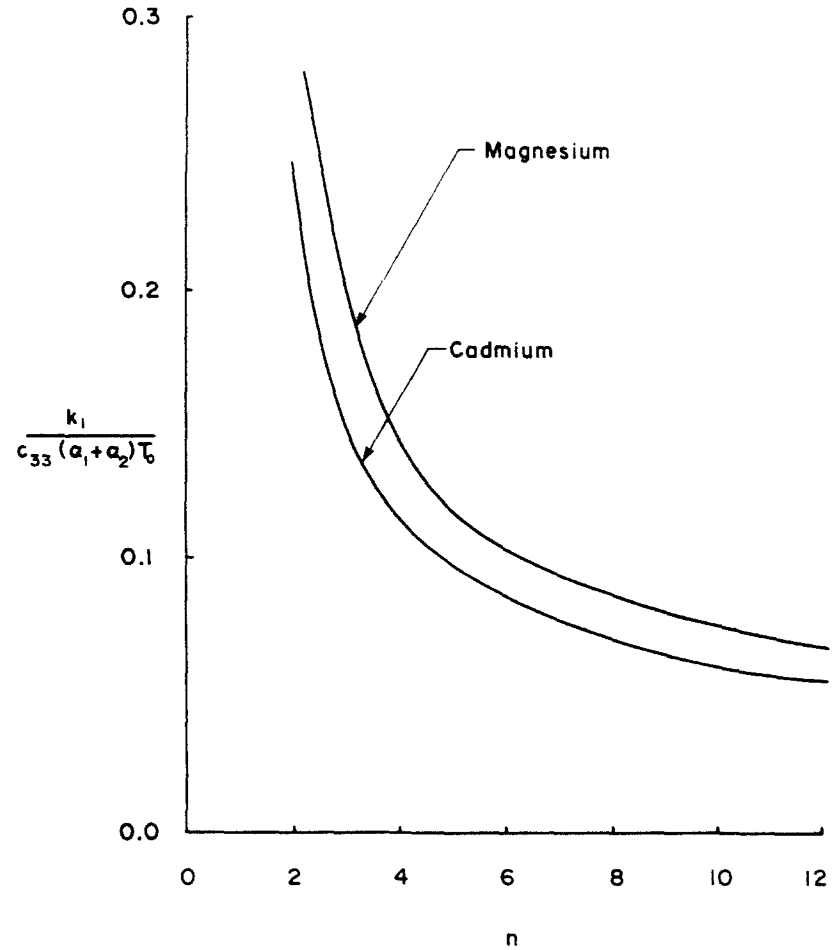

Fig. 2. Variation of stress intensity factor with temperature.

REFERENCES

1. Z. Olesiak and I. N. Sneddon, The distribution of thermal stress in an infinite elastic solid containing a penny-shaped crack. *Archs Ration. Mech. Analysis* **4**, 238–254 (1960).
2. A. L. Florence and J. N. Goodier, The linear thermoelastic problem of uniform heat flow distributed by a penny-shaped crack. *Int. J. Engng Sci.* **1**, 533–540 (1963).
3. M. K. Kassir and G. C. Sih, Thermal stresses in solid weakened by an external circular crack. *Int. J. Solids Structures* **5**, 351–367 (1969).
4. M. K. Kassir, Size of thermal plastic zones around external cracks. *Int. J. Fracture Mech.* **5**, 167–177 (1969).
5. Y. M. Tsai, Thermal stress in a transversely isotropic medium containing a penny-shaped crack. *J. Appl. Mech.* **50**, 24–28 (1983).
6. Y. M. Tsai, Penny-shaped crack in a transversely isotropic-plate of finite thickness. *Int. J. Fracture Mech.* **20**, 81–89 (1982).
7. M. Lowengrub and I. N. Sneddon, The solution of pair of dual integral equations. *Proc. Glasgow Math. Ass.* **6**, 14–18 (1963).
8. B. Noble, The solution of Bessel function dual integral equations by multiplying-factor method. *Proc. Camb. Phil. Soc.* **59**, 351–362 (1963).
9. B. Sharma and M. Pradesh, Thermal stresses in transversely isotropic semi-infinite elastic solid. *ASME, J. Appl. Mech.* **25**, 86–88 (1958).

APPENDIX

For determining the displacements and stresses, let us take

$$u(r, z) = \frac{\partial}{\partial r}(\phi + \mu_1\psi) \tag{43}$$

$$w(r, z) = \frac{\partial}{\partial z}(\lambda\phi + \mu_2\psi) \tag{44}$$

where ϕ and ψ are functions of r and z only, λ, μ_1 and μ_2 are material constants. Now substituting eqns (43) and (44) into eqns (2) and (3) will be satisfied if

$$c_{11}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)+(c_{44}+\lambda(c_{13}+c_{44}))\frac{\partial^2\phi}{\partial z^2}=0 \tag{45}$$

$$\mu_1 c_{11}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+(\mu_1 c_{44}+\mu_2(c_{13}+c_{44}))\frac{\partial^2\psi}{\partial z^2}=b_1 T \tag{46}$$

$$(c_{13}+c_{44}+\lambda c_{44})\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)+\lambda c_{33}\frac{\partial^2\phi}{\partial z^2}=0 \tag{47}$$

$$[\mu_1(c_{13}+c_{44})+\mu_2 c_{44}]\left[\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right]+\mu_2 c_{33}\frac{\partial^2\psi}{\partial z^2}=b_2 T. \tag{48}$$

To find the solution of eqns (46) and (48) we assume that

$$\psi(r,z)=\int_0^\infty \xi^{-2}A(\xi)\mathrm{e}^{-\xi z/\beta}J_0(\xi r)\,\mathrm{d}\xi. \tag{49}$$

Making use of eqns (46) and (48) we find from eqns (7) and (49) that

$$\mu_1(c_{44}-\beta^2 c_{11})+\mu_2(c_{13}+c_{44})=b_1\beta^2 \tag{50}$$

$$\mu_2(c_{33}-c_{44}\beta^2)-\mu_1(c_{13}+c_{44})\beta^2=b_2\beta^2. \tag{51}$$

From eqns (50) and (51) we get

$$\mu_1=\frac{\beta^2[b_1(c_{33}-\beta^2 c_{44})-b_2(c_{13}+c_{44})]}{(c_{44}-\beta^2 c_{11})(c_{33}-\beta^2 c_{44})+\beta^2(c_{13}+c_{44})^2} \tag{52}$$

$$\mu_2=\frac{\beta^2[b_1(c_{13}+c_{44})\beta^2+b_2(c_{44}-\beta^2 c_{11})]}{(c_{44}-\beta^2 c_{11})(c_{33}-\beta^2 c_{44})+\beta^2(c_{13}+c_{44})^2}. \tag{53}$$

Equations (50) and (51) will give non-trivial solutions provided

$$\frac{\lambda(c_{13}+c_{44})+c_{44}}{c_{11}}=\frac{\lambda c_{33}}{\lambda c_{44}+c_{13}+c_{44}}=m^2\,(\text{say})$$

or

$$c_{11}c_{44}m^4+(c_{13}^2+2c_{13}c_{44}-c_{11}c_{33})m^2+c_{33}c_{44}=0. \tag{54}$$

The solutions of equations of equilibrium involving $\phi(r,z)$ given by eqns (45) and (47) can be easily found in terms of two stress functions $\phi_1(r,z)$ and $\phi_2(r,z)$ and the expressions for the stresses and the displacements can be obtained from the results given by Sharma and Pradesh[9] in the following form:

$$\begin{aligned}
\sigma_{zz}&=c_{13}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(\phi_1+\phi_2+\mu_1\psi)+c_{33}\frac{\partial^2}{\partial z^2}(\lambda_1\phi_1+\lambda_2\phi_2+\mu_2\psi)-b_2 T\\
\sigma_{rz}&=c_{44}\frac{\partial^2}{\partial r\partial z}[(1+\lambda_1)\phi_1+(1+\lambda_2)\phi_2+(\mu_1+\mu_2)\psi]\\
u&=\frac{\partial}{\partial r}(\phi_1+\phi_2+\mu_1\psi)\\
w&=\frac{\partial}{\partial z}(\lambda_1\phi_1+\lambda_2\phi_2+\mu_2\psi)
\end{aligned} \tag{55}$$

where $\phi_1(r,z)$ and $\phi_2(r,z)$ are the solutions of the partial differential equations

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+m_i^2\frac{\partial^2}{\partial z^2}\right)\phi_i=0,\qquad i=1,2 \tag{56}$$

where m_1^2 and m_1^2 are the roots of the quadratic eqn (54), and λ_1 and λ_2 are two values of λ corresponding to m_2^2 and m_2^2, respectively. Solution of eqn (56) can be written as

$$\phi_1(r, z) = \int_0^{\infty} B(\xi)\, e^{-\xi z/m_1} J_0(\xi r)\, d\xi \tag{57}$$

$$\phi_2(r, z) = \int_0^{\infty} C(\xi)\, e^{-\xi z/m_2} J_0(\xi r)\, d\xi \tag{58}$$

where $B(\xi)$ and $C(\xi)$ are both unknown functions to be determined from the elastic boundary conditions (6). Inserting the values $\phi_1(r, z)$, $\phi_2(r, z)$, $T(r, z)$ and $\psi(r, z)$ from eqns (57), (58), (7) and (49) in eqn (55) we obtain the displacements and stresses at any point in the form given in eqn (15).